THE FUNDAMENTAL THEOREM OF ALGEBRA: FROM THE FOUR BASIC OPERATIONS

Oswaldo Rio Branco de Oliveira

Abstract. This paper presents an elementary and direct proof of the Fundamental Theorem of Algebra, via Bolzano-Weierstrass Theorem on Minima, that avoids: every root extraction, angle, non-algebraic functions , differentiation, integration, series and arguments by induction.

Mathematics Subject Classification: 12D05, 30A10

Key words and phrases: Fundamental Theorem of Algebra, Inequalities in $\mathbb{C}$.

This article aims, by combining an inequality proved in Oliveira [10] and a lemma by Estermann [5], to show a most elementary proof of the FTA that does not use any root extraction. Following a suggestion given by Littlewood [9], see also Remmert [12], the proof requires a minimum amount of "limit processes lying outside algebra proper". Hence, the proof avoids differentiation, integration, series, angle and the transcendental functions (i.e., non-algebraic functions) $\cos\theta$, $\sin\theta$ and $e^{i\theta}$, $\theta \in \mathbb{R}$. Another reason to avoid these functions is justified by the fact that the theory of transcendental functions is more profound than that of the FTA (a polynomial result), see Burckel [4]. It is good to notice that the usual proof of the well known Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, $\theta \in \mathbb{R}$, requires series, differentiation and the (transcendental) numbers e and π (see Rudin [13]). Also avoided are arguments by induction and $\epsilon - \delta$ type arguments.

Many elementary proofs of the FTA, implicitly assuming the modulus function $|z| = \sqrt{z\overline{z}}$, where $z \in \mathbb{C}$, assume either the Bolzano-Weierstrass Theorem on Minima or the Intermediate Value Theorem, plus polynomial continuity. Then, along the proof, it is used further root extraction in $\mathbb{R}$ or in $\mathbb{C}$ (see Aigner and Ziegler [1], Argand [2] and [3], Estermann [5], Fefferman [6], Kochol [8], Littlewood [9], Oliveira [10], Redheffer [11], Remmert [12], Rudin [13], Searcóid [15], Spivak [16], Terkelsen [17]. See also, Schep [14], Vaggione [18] and [19] and Výborný [20]. Beginning with Littlewood [9],

some of these proofs include a proof by induction of the existence of every nth root, $n \in \mathbb{N}$, of every complex number (see [8], [12], [15] and [16]).

It is well known that all norms over $\mathbb{C}$ are equivalent and that $\mathbb{C}$, equipped with any norm, is complete. In what follows it is considered the norm $|z|_1 = |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$, $z \in \mathbb{C}$. Given $z, w \in \mathbb{C}$, it is easy to see that

$$|\overline{z}|_1 = |z|_1 \quad \text{and} \quad \frac{|z|_1\,|w|_1}{2} \leq |zw|_1 \leq |z|_1\,|w|_1 \ .$$

Moreover, in what follows it will be needed the well known Binomial Formula $(z+w)^n = \sum_{j=0}^{n} \binom{n}{j} z^j w^{n-j}$, $z \in \mathbb{C}$, $w \in \mathbb{C}$, $n \in \mathbb{N}$, $\binom{n}{j} = \frac{n!}{j!(n-j)!}$ and $0! = 1$. It is assumed, without proof, only:

- Polynomial continuity.
- Bolzano-Weierstrass Theorem: *Any continuous function $f : D \to \mathbb{R}$, D a bounded and closed disc, has a minimum on D.*

Right below, we show, for the case k even, $k \geq 2$, a pair of inequalities that Estermann [5] proved for every $k \in \mathbb{N} \setminus \{0\}$. The proof, via binomial formula, simplifies Estermann's proof, which uses root extraction and also induction. The case k odd can be proved similarly, if one wishes.

Lemma (Estermann). For $\zeta = \left(1 + \frac{i}{k}\right)^2$ and k even, $k \geq 2$, we have

$$\mathrm{Re}[\zeta^k] \ < \ 0 \ < \ \mathrm{Im}[\zeta^k] \ .$$

Proof. Since $k = 2m$ and $2k = 4m$, for some $m \in \mathbb{N}$, applying the formulas

$$\mathrm{Re}\left[\left(1 + \tfrac{i}{k}\right)^{2k}\right] = 1 - \binom{2k}{2}\frac{1}{k^2} + \binom{2k}{4}\frac{1}{k^4} + \sum_{\text{odd } j\,, j=3}^{k-1} \left[-\binom{2k}{2j}\frac{1}{k^{2j}} + \binom{2k}{2j+2}\frac{1}{k^{2j+2}}\right] \ \text{and}\,,$$

$$\mathrm{Im}\left[\left(1 + \tfrac{i}{k}\right)^{2k}\right] = \sum_{\text{odd } j\,, j=1}^{k-1} \left[\binom{2k}{2j-1}\frac{1}{k^{2j-1}} - \binom{2k}{2j+1}\frac{1}{k^{2j+1}}\right] \ ,$$

we end the proof by noticing that for every $j \in \mathbb{N}$, $1 \leq j \leq k-1$, we have

$$\begin{aligned} 1 - \binom{2k}{2}\frac{1}{k^2} + \binom{2k}{4}\frac{1}{k^4} \quad &= 1 - \left(2 - \frac{1}{k}\right)\left(\frac{2}{3} + \frac{5}{6k} - \frac{1}{2k^2}\right) \leq \\ &\leq 1 - \frac{3}{2}\left(\frac{2}{3} + \frac{5k-3}{6k^2}\right) = -\frac{3}{2} \cdot \frac{5k-3}{6k^2} < 0\,, \end{aligned}$$

$$-\binom{2k}{2j}\frac{1}{k^{2j}} + \binom{2k}{2j+2}\frac{1}{k^{2j+2}} \quad = -\frac{(2k)!}{(2j)!\,k^{2j}\,(2k-2j-2)!}\left[\frac{1}{(2k-2j)(2k-2j-1)} - \frac{1}{(2kj+2k)(2kj+k)}\right] < 0\,,$$

$$\binom{2k}{2j-1}\frac{1}{k^{2j-1}} - \binom{2k}{2j+1}\frac{1}{k^{2j+1}} \quad = \frac{(2k)!}{(2j-1)!\,(2k-2j-1)!}\frac{1}{k^{2j-1}}\left[\frac{1}{(2k-2j+1)(2k-2j)} - \frac{1}{(2kj+k)(2kj)}\right] > 0 \ .$$

Theorem. Let P be a complex polynomial, with degree$(P) = n \geq 1$. Then, P has a zero.

Proof. Putting $P(z) = a_0 + a_1 z + ... + a_n z^n$, $a_j \in \mathbb{C}$, $0 \leq j \leq n$, $a_n \neq 0$, we have

$$P(z)\overline{P(z)} = \sum_{j=0}^{n} a_j \overline{a_j} z^j \overline{z}^j \ + \sum_{0 \leq j < k \leq n} 2\mathrm{Re}[a_j \overline{a_k} z^j \overline{z}^k] \,, \ \forall z \in \mathbb{C} \,.$$

Clearly, $P(z)\overline{P(z)} \geq |a_n|_1^2 |z|_1^{2n}/2^{2n+1} - \sum_{0 \leq j < k \leq n} 2|a_j|_1 |a_k|_1 |z|_1^{j+k}, \forall z \in \mathbb{C}$. Hence, $P(z)\overline{P(z)} \to \infty$ as $|z|_1 \to \infty$ and, by continuity, $P\overline{P}$ has a global minimum at some $z_0 \in \mathbb{C}$. We can clearly assume that $z_0 = 0$. Therefore,

$$(1) \qquad P(z)\overline{P(z)} - P(0)\overline{P(0)} \geq 0 \,, \ \forall \, z \in \mathbb{C} \,,$$

and $P(z) = P(0) + z^k Q(z)$, for some $k \in \{1, ..., n\}$, where Q is a polynomial and $Q(0) \neq 0$. Substituting this equation, at $z = r\zeta$, where $r \geq 0$ and ζ is arbitrary in $\mathbb{C}$, in inequality (1), we arrive at

$$2r^k \mathrm{Re}\big[\, \overline{P(0)} \zeta^k Q(r\zeta) \,\big] \ + \ r^{2k} \zeta^k Q(r\zeta) \overline{\zeta^k Q(r\zeta)} \ \geq \ 0 \,, \ \forall r \geq 0 \,, \ \forall \zeta \in \mathbb{C},$$

and, cancelling $r^k > 0$, we find the inequality

$$2\mathrm{Re}\big[\, \overline{P(0)} \zeta^k Q(r\zeta) \,\big] + r^k \zeta^k Q(r\zeta) \overline{\zeta^k Q(r\zeta)} \ \geq \ 0 \,, \forall \, r > 0 \,, \forall \, \zeta \in \mathbb{C} \,,$$

whose left side is a continuous function of r, $r \in [0, +\infty)$. Thus, taking the limit as $r \to 0$ we find,

$$(2) \qquad 2\mathrm{Re}\big[\, \overline{P(0)} Q(0) \zeta^k \,\big] \geq 0 \,, \ \forall \, \zeta \in \mathbb{C} \,.$$

Let $\alpha = \overline{P(0)} Q(0) = a + ib$, where $a, b \in \mathbb{R}$. If k is odd then, substituting $\zeta = \pm 1$ and $\zeta = \pm i$ in (2), we reach $a = 0$ and $b = 0$. Hence $\alpha = 0$ and then, $P(0) = 0$. Thus, the case k odd is proved. Next, let us suppose k even. Taking $\zeta = 1$ in (2), we conclude that $a \geq 0$. Picking ζ as in the lemma, let us write $\zeta^k = x + iy$, with $x < 0$ and $y > 0$. Substituting ζ^k and $\overline{\zeta}^k = \overline{\zeta^k}$ in (2) we arrive at $\mathrm{Re}[\alpha(x \pm iy)] = ax \mp by \geq 0$. Hence $ax \geq 0$ and (since $x < 0$) we conclude that $a \leq 0$. So, $a = 0$. Therefore, we get $\mp by \geq 0$. Hence, since $y \neq 0$, we conclude that $b = 0$. Hence $\alpha = 0$ and then, $P(0) = 0$. Thus, the case k even is proved. The theorem is proved.

Remarks

(1) By equipping $\mathbb{C}$ with the usual norm $|z| = \sqrt{z\bar{z}}$, one can easily adapt the proof above to produce a "more familiar" and easier to follow proof of the FTA, at the cost of introducing the square root function in the proof. In such case, the inequality $|P(z)| \geq |a_n||z|^n - \sum_{j=0}^{n-1} |a_j||z|^j$ implies that the function $|P|$ has a global minimum at some $z_0 \in \mathbb{C}$. Then, supposing without loss of generality $z_0 = 0$, one can analize the inequality $|P(z)|^2 - |P(0)|^2 \geq 0$ exactly as it was done above.

(2) The almost algebraic "Gauss' Second Proof" (see [7]) of the FTA uses only that "every real polynomial of odd degree has a real zero" and the existence of a positive square root of every positive real number. Nevertheless, this proof by Gauss is not elementary.

(3) It is possible to rewrite a small part of the given proof of the FTA so that the polynomial continuity is used only to guarantee the existence of z_0, a point of global minimum of $|P|$. In fact, to avoid extra use of polynomial continuity, let us keep the notation of the proof and indicate $Q(z) = Q(0) + zR(z)$, with R a polynomial. Then, substituting this expression for $Q(z)$, at $z = r\zeta$, only in the first parcel in the left side of the inequality $2\text{Re}\big[\,\overline{P(0)}\zeta^k Q(r\zeta)\,\big] + r^k\zeta^k Q(r\zeta)\overline{\zeta^k Q(r\zeta)} \geq 0\,,\ \forall r > 0\,,\ \forall \zeta \in \mathbb{C}$, that appeared just above inequality (2), we get

$$2\text{Re}\big[\,\overline{P(0)}\zeta^{\,k}Q(0)\,\big] + 2r\text{Re}\big[\,\overline{P(0)}\zeta^{k+1}R(r\zeta)\,\big] + r^k\zeta^k Q(r\zeta)\overline{\zeta^k Q(r\zeta)} \geq 0\,,$$

$\forall\, r > 0\,, \forall\, \zeta \in \mathbb{C}$. Fixing ζ arbitrary in $\mathbb{C}$, it is clear that there is a constant $M = M(\zeta) > 0$ such that the following inequality is satisfied: $\max\big(\,|P(0)\zeta^{k+1}R(r\zeta)|_1\,, |\zeta^k Q(r\zeta)|_1^2\,\big) \leq M$, $\forall r \in (0,1)$. Hence,

$$-2\text{Re}\big[\,\overline{P(0)}\zeta^{\,k}Q(0)\,\big] \leq 2rM + r^k M \leq 3rM\,,\ \ \forall\, r \in (0,1).$$

So, we conclude that $-2\text{Re}\big[\,\overline{P(0)}\zeta^{\,k}Q(0)\,\big] \leq 0$, with ζ arbitrary in $\mathbb{C}$. Now, obviously, the proof continues as in the theorem proof.

(4) It is worth to point out that this proof of the FTA easily implies an independent proof of the existence of a unique positive nth root, $n \geq 2$,

of each positive number c. To show this, let us fix $c \geq 0$. Considering $n = 2$, and applying the FTA, we can pick $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, such that $c = z^2 = (x^2 - y^2) + 2xyi$. Hence, we have that $y = 0$ and $x^2 = c$. So, $(\pm x)^2 = c$. Let $\sqrt{c}$ be the unique positive square root of c. Hence, it is well defined the absolute value $|z| = \sqrt{z\overline{z}}$. Lastly, given an arbitrary $n \in \mathbb{N}$, $n \geq 2$, let us pick $z \in \mathbb{C}$ such that $z^n = \sqrt{c}$. Therefore, we have that $z^{2n} = c$ and, by the well known properties of the absolute value function over $\mathbb{C}$, $(|z|^2)^n = |z^{2n}| = c$. The uniqueness of a positive nth root of c is trivial.

Acknowledgments

I thank Professors J. V. Ralston and Paulo A. Martin for their very valuable comments and suggestions, J. Aragona for reference [6] and R. B. Burckel for references [8], [18] and [19], besides his article [4].

References

[1] Aigner, M. and Ziegler, G. M., *Proofs from THE BOOK*, 4th edition, Springer, 2010.

[2] Argand, J. R., *Imaginary Quantities: Their Geometrical Interpretation*, University Michigan Library, 2009.

[3] Argand, J. R., *Essay Sur Une Manière de Représenter Les Quantités Imaginaires Dans Les Contructions Géométriques*, Nabu Press, 2010.

[4] Burckel, R. B., " A Classical Proof of The Fundamental Theorem of Algebra Dissected", *Mathematical Newsletter of the Ramanujan Mathematical Society 7*, no. 2 (2007), 37-39.

[5] Estermann, T., "On The Fundamental Theorem of Algebra", *J. London Mathematical Society* 31 (1956), 238-240.

[6] Fefferman, C., "An Easy Proof of the Fundamental Theorem of Algebra", *American Mathematical Monthly* 74 (1967), 854-855.

[7] Gauss, C. F., *Werke*, Volume 3, 33-56 (in latin; English translation available at http:www.cs.man.ac.uk/∼pt/misc/gauss-web.html).

[8] Kochol, M., "An Elementary Proof of The Fundamental Theorem of Algebra", International Journal of Mathematical Education in Science and Technology, 30 (1999), 614-615.

[9] Littlewood, J. E., "Mathematical notes (14): Every Polynomial has a Root", *J. London Mathematical Society* 16 (1941), 95-98.

[10] Oliveira, O. R. B., "The Fundamental Theorem of Algebra: An Elementary and Direct Proof", *The Mathematical Intelligencer* 33, No. 2, (2011), 1-2. In http://www.springerlink.com/content/l1847265q2311325

[11] Redheffer, R. M., "What! Another Note Just on the Fundamental Theorem of Algebra?", *American Mathematical Monthly* 71 (1964), 180-185.

[12] Remmert, R., "The Fundamental Theorem of Algebra", in H.-D. Ebbinghaus, et al., *Numbers*, Graduate Texts in Mathematics, no. 123, Springer-Verlag, New York, 1991. Chapters 3 and 4.

[13] Rudin, W., *Principles of Mathematical Analysis*, McGraw-Hill Inc., Tokio, 1963

[14] Schep, A. R., "A Simple Complex Analysis and an Advanced Calculus Proof of the Fundamental Theorem of Algebra", *The American Mathematical Monthly*, 116 (January, 2009), 67-68

[15] Searcóid, M. O., *Elements of Abstract Analysis*, Springer-Verlag, London, 2003.

[16] Spivak, M., *Calculus*, 4th edition, Publish or Perish, Inc., 2008.

[17] Terkelsen, F., "The Fundamental Theorem of Algebra", *The American Mathematical Monthly*, Vol. 83, No. 8 (October, 1976), p. 647.

[18] Vaggione, D., "On The Fundamental Theorem of Algebra", *Colloquium Mathematicum* 73 No. 2 (1997), 193-194.

[19] Vaggione, D., Errata to "On The Fundamental Theorem of Algebra", *Colloquium Mathematicum* 73 (1998), p. 321.

[20] Výborný, R., "A simple proof of the Fundamental Theorem of Algebra", *Mathematica Bohemica*, Vol 135, No. 1 (2010), 57-61.

Departamento de Matemática
Universidade de São Paulo
Rua do Matão 1010, CEP 05508-090
São Paulo, SP
Brasil
e-mail: oliveira@ime.usp.br